

# Discrete-time Quantum Walks in random artificial Gauge Fields

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Discrete-time quantum walks (DTQWs) in random artificial electric and gravitational fields are studied analytically and numerically. The analytical computations are carried by a new method which allows a direct exact analytical determination of the equations of motion obeyed by the average density operator. It is proven that randomness induces decoherence and that the quantum walks behave asymptotically like classical random walks. Asymptotic diffusion coefficients are computed exactly. The continuous limit is also obtained and discussed.

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## I. INTRODUCTION

Discrete time quantum walks (DTQWs) are simple formal analogues of classical random walks. They were first considered by Feynmann in [1], and then introduced in greater generality in [2] and [3]. They have been realized experimentally [4–10] and are important in many fields, ranging from fundamental quantum physics [10, 11] to quantum algorithmics [12, 13], solid state physics [14–17] and biophysics [18, 19].

It has been shown [20–22] recently that several DTQWs on the line admit a continuous limit identical to the propagation of a Dirac fermion in artificial electric and gravitational fields. These DTQWs are thus simple discrete models of quantum propagation in artificial gauge fields. Here, we consider artificial gauge fields which depend randomly on time and investigate analytically and numerically how this randomness influences quantum propagation. The analysis presented in this article is based on a direct analytical computation of the exact evolution equation obeyed by the average density operator. This presents several advantages. First, the average dynamics is thus known exactly, without the noise inherent in any numerical evaluation of averages. Second, knowing the exact average equations of motion makes it possible to study the average dynamics analytically. Finally, simulating directly the exact analytical equations of the average dynamics offers a significant gain in computation time over alternative methods where the average evolution is determined by simulating successively a large number of realizations of the random DTQWs.

Random DTQWs have already been studied by several authors (see for example [23–28]), but the influence of random gauge fields has never been the object of specific analytical computations. In particular, exact expressions of the asymptotic density profiles as functions of the randomness characteristics have never been computed. Our main results are (i) DTQWs interacting with artificial gauge fields which are random in time decohere and behave asymptotically like classical random walks (ii) the asymptotic density profiles of the DTQWs are Gaussian and we give exact analytical expressions of the asymptotic diffusion coefficients as functions of the noise amplitude which generates the randomness. We also support all results by direct numerical simulations of the average dynamics and finally discuss the continuous limits of the DTQWs interacting with random artificial gauge fields.

## II. A FAMILY OF DTQWS COUPLED TO ARTIFICIAL ELECTRIC AND GRAVITATIONAL FIELDS

### A. Wave-function evolution

#### 1. In physical space

We consider discrete time quantum walks in one space dimension driven by a time-dependent quantum coin acting on a two-dimensional Hilbert space  $\mathcal{H}$ . The walks are defined by the following finite difference equations, valid for all  $(j, m) \in \mathbb{N} \times \mathbb{Z}$ :

$$\begin{bmatrix} \psi_{j+1,m}^L \\ \psi_{j+1,m}^R \end{bmatrix} = \mathcal{B}(\theta_j, \xi_j) \begin{bmatrix} \psi_{j,m+1}^L \\ \psi_{j,m-1}^R \end{bmatrix}, \quad (1)$$

where

$$\mathcal{B}(\theta, \xi) = \begin{bmatrix} e^{i\xi} \cos \theta & i \sin \theta \\ i \sin \theta & e^{-i\xi} \cos \theta \end{bmatrix}. \quad (2)$$

The operator represented by the matrix  $\mathcal{B}$  is in  $SU(2)$  and  $\theta$  and  $\xi$  are two of the three Euler angles. The index  $j$  labels instants and takes all positive integer values. The index  $m$  labels spatial points. We choose to work on the circle and impose periodic boundary conditions. We thus introduce a strictly positive integer  $M$  and restrict  $m$  to all integer values between  $-M$  and  $+M$  i.e.  $m \in \mathbb{Z}_M$ . Results pertaining to DTQWs on the infinite line can be recovered by letting  $M$  tend to infinity.

For each instant  $j$  and each spatial point  $m$ , the wave function  $\Psi_{jm} = \psi_{jm}^L b_L + \psi_{jm}^R b_R = \psi_{jm}^a b_a, a \in \{L, R\}$ , has two components  $\psi_{jm}^L$  and  $\psi_{jm}^R$  on the spin basis  $(b_L, b_R)$  and these code for the probability amplitudes of the particle jumping towards the left or towards the right. Note that the spin basis is interpreted as being independent of  $j$  and  $m$ . For a given initial condition, the set of angles  $\{\theta_j, \xi_j, j \in \mathbb{N}\}$  completely defines the walks and is arbitrary.

It has been proven in [20–22] that walks from this family are models of Dirac fermions coupled to artificial electric and gravitational fields. Details can be found in these references and in the first appendix to the present article.

## 2. In Fourier space

A practical tool to study quantum walks on the discrete circle is the discrete Fourier transform (DFT). Let  $(A_m)_{m \in \mathbb{Z}_M}$  be an arbitray sequence of complex numbers defined on the discrete circle. The DFT of this sequence is the sequence  $(\hat{A}_{k_n})_{n \in \mathbb{Z}_M}$  defined by

$$\hat{A}_{k_n} = \sum_{m=-M}^{+M} A_m \exp(ik_n m) \quad (3)$$

with  $k_n = 2n\pi/(2M+1)$ ,  $n \in \mathbb{Z}_M$ . The original sequence can be recovered from its DFT by the relation:

$$A_m = \frac{1}{2M+1} \sum_{n=-M}^{+M} \hat{A}_{k_n} \exp(-ik_n m). \quad (4)$$

For infinite  $M$  i.e. DTQWs on the infinite line, the DFT of an infinite sequence  $(A_m)_{m \in \mathbb{Z}}$  becomes a function

$$\hat{A}(k) = \sum_{m \in \mathbb{Z}} A_m \exp(ikm) \quad (5)$$

defined for  $k \in (-\pi, \pi)$  and the inverse relation reads:

$$A_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{A}(k) \exp(-ikm) dk. \quad (6)$$

In Fourier space on the infinite line, the evolution equation (1) transcribes into

$$\begin{bmatrix} \hat{\psi}_{j+1}^L(k) \\ \hat{\psi}_{j+1}^R(k) \end{bmatrix} = \mathcal{C}(\theta_j, \xi_j, k) \begin{bmatrix} \hat{\psi}_j^L(k) \\ \hat{\psi}_j^R(k) \end{bmatrix} \quad (7)$$

where

$$\mathcal{C}(\theta_j, \xi_j, k) = \begin{bmatrix} e^{i\xi} \cos \theta e^{-ik} & i \sin \theta e^{+ik} \\ i \sin \theta e^{-ik} & e^{-i\xi} \cos \theta e^{+ik} \end{bmatrix} \quad (8)$$

for all  $k \in (-\pi, \pi)$ .

## B. Density operator evolution

### 1. In physical space

The walks can also be described using the density operator  $\rho = \Psi^* \otimes \Psi$ . We introduce the basis  $v_1 = b_L \otimes b_L$ ,  $v_2 = b_L \otimes b_R$ ,  $v_3 = b_R \otimes b_L$ ,  $v_4 = b_R \otimes b_R$  and represent  $\rho$  by its components on this basis i.e. by the quantities  $\rho_{j,m,m'}^{ab} = \psi_{jm}^{b*} \psi_{jm'}^a$ ,  $\{a, b\} \in \{L, R\}^2$ . Equation (1) delivers:

$$\begin{bmatrix} \rho_{j+1,m,m'}^{LL} \\ \rho_{j+1,m,m'}^{LR} \\ \rho_{j+1,m,m'}^{RL} \\ \rho_{j+1,m,m'}^{RR} \end{bmatrix} = \mathcal{Q}(\theta_j, \xi_j) \begin{bmatrix} \rho_{j,m+1,m'+1}^{LL} \\ \rho_{j,m+1,m'-1}^{LR} \\ \rho_{j,m-1,m'+1}^{RL} \\ \rho_{j,m-1,m'-1}^{RR} \end{bmatrix} \quad (9)$$

where

$$\mathcal{Q}(\theta, \xi) = \begin{bmatrix} c^2 & -ics e^{+i\xi} & +ics e^{-i\xi} & s^2 \\ -ics e^{+i\xi} & c^2 e^{+2i\xi} & s^2 & +ics e^{+i\xi} \\ +ics e^{-i\xi} & s^2 & c^2 e^{-2i\xi} & -ics e^{-i\xi} \\ s^2 & +ics e^{+i\xi} & -ics e^{-i\xi} & c^2 \end{bmatrix}, \quad (10)$$

with  $c = \cos \theta$  and  $s = \sin \theta$ . The probability to find the walk at time  $j$  at point  $m$  is  $N_{jm} = \rho_{j,m,m}^{LL} + \rho_{j,m,m}^{RR}$  and the sum  $\sum_m N_{jm}$  is independent of  $j$  i.e. it is conserved by the walk. Contrary to equation (1), equation (9) can be used to describe walks with initial conditions which are not pure states. Equation (9) is thus more general than (1).

## 2. In Fourier space

Consider now, for any instant  $j$ , the double DFT of the density operator  $\rho_{j,m,m'}$ , which we denote by  $\hat{\rho}_j(k, k')$  or, alternately,  $\hat{\rho}_j(K, p)$  where  $K = (k + k')/2$  is conjugate to  $m + m'$  and  $p = (k' - k)/2$  is conjugate to  $m' - m$ . For DTQWs on the infinite line, the range of both  $K$  and  $p$  is  $(-\pi, +\pi)$ . The DFT of the density operator obeys  $\hat{\rho}_{j+1}(K, p) = \mathcal{R}(\theta_j, \xi_j, K, p) \hat{\rho}_j(K, p)$  with

$$\mathcal{R}(\theta, \xi, K, p) = \begin{bmatrix} c^2 e^{2iK} & -ics e^{+i\xi} e^{-2ip} & +ics e^{-i\xi} e^{+2ip} & s^2 e^{-2iK} \\ -ics e^{+i\xi} e^{2iK} & c^2 e^{+2i\xi} e^{-2ip} & s^2 e^{+2ip} & +ics e^{+i\xi} e^{-2iK} \\ +ics e^{-i\xi} e^{2iK} & s^2 e^{-2ip} & c^2 e^{-2i\xi} e^{+2ip} & -ics e^{-i\xi} e^{-2iK} \\ s^2 e^{2iK} & +ics e^{+i\xi} e^{-2ip} & -ics e^{-i\xi} e^{+2ip} & c^2 e^{-2iK} \end{bmatrix}. \quad (11)$$

Note that the operator  $\mathcal{R}$  governing the evolution of  $\bar{\rho}$  is unitary. This can be checked by a straightforward computation and it is a direct consequence of the unitarity of the operator  $\mathcal{B}$ .

## III. RANDOMIZING THE FIELDS AND AVERAGING THE DYNAMICS

### A. Randomizing the fields

The Hadamard walk corresponds to  $\xi = \xi_H = \pi/2$  and  $\theta = \theta_H = \pi/4$ ; since these angles are constant, the Hadamard walk describes propagation in the absence of electric and gravitational field [21, 22]. We now consider situations where one of the angles  $\xi$  and  $\theta$  does depend on time and fluctuates around its Hadamard value. More precisely, we consider two cases. Case 1 corresponds to  $\theta = \theta_H = \pi/4$  and  $\xi$  chosen randomly at each time-step with uniform probability law in the interval  $(\pi/2 - \sigma/2, \pi/2 + \sigma/2)$ , where  $\sigma \in (0, 2\pi)$  is a fixed i.e.  $j$ -independent positive real number. As proven in [20–22] and detailed in the first appendix to the present article, a time-dependent  $\theta$  is equivalent to a space-time metric whose purely spatial part depends on time, and such a metric represents a time-dependent relativistic gravitational field. Case 2 corresponds to  $\xi = \xi_H = \pi/2$  and  $\theta$  chosen randomly at each time-step with uniform probability law in the interval  $(\pi/4 - \sigma/2, \pi/4 + \sigma/2)$ . As proven in [22], a time-dependent  $\xi$  is equivalent to a time-dependent ‘vector’ potential, which represents a time-dependent electric field.

Thus, in each case, a realization of the random gauge field is determined by a sequence  $\omega = (\omega_1, \omega_2, \dots)$  of independent random variables, where  $\omega_j$  represents the value of the random angle  $\theta$  or  $\xi$  at time  $j$ . If one follows the walk till time  $N$ , the relevant random sequence is the  $N$ -uple  $\omega^N = (\omega_1, \omega_2, \dots, \omega_N)$ . For each value of  $\sigma$  and each instant  $j$ ,  $\omega_j$  is uniformly distributed in the interval  $I_\sigma = (\omega_H - \sigma/2, \omega_H + \sigma/2)$  centered on the Hadamard value  $\omega_H$ . The probability density of  $\omega_j$  in this interval is thus simply  $p_\sigma(\omega_j) = 1/\sigma$  and is independent of both  $\omega_j$  and  $j$ . The probability density for  $\omega^N = (\omega_1, \omega_2, \dots, \omega_N)$  in  $I_\sigma^N$  is therefore  $P_\sigma(\omega^N) = \prod_{j=1}^N p_\sigma(\omega_j) = 1/\sigma^N$  and is independent of  $\omega^N$ .

### B. Averaging the dynamics

At fixed initial condition  $\rho_0$  and for each time  $N$ , the density operator  $\rho_N$  at time  $N$  depends on the realization  $\omega^N$  of the random angle up to time  $N$ . At fixed initial condition, the easiest way to compute statistical averages over

$\omega^N$  is to first compute the statistical average  $\bar{\rho}_N$  of the density operator over  $\omega^N$ :

$$\begin{aligned}\bar{\rho}_N &= \int_{I_\sigma^N} \rho_N(\omega^N) P_\sigma(\omega^N) d\omega^N \\ &= \int_{I_\sigma^N} \rho_N(\omega_1, \dots, \omega_N) p_\sigma(\omega_1) \dots p_\sigma(\omega_N) d\omega_1 \dots d\omega_N \\ &= \int_{I_\sigma^N} \rho_N(\omega_1, \dots, \omega_N) \frac{1}{\sigma^N} d\omega_1 \dots d\omega_N.\end{aligned}\quad (12)$$

Let us work in Fourier space. One can then write, for any realization  $\omega^N = (\omega_1, \dots, \omega_N)$  of the random angle up to time  $N$ :

$$\begin{aligned}\hat{\rho}_N(\omega_1, \dots, \omega_N) &= \mathcal{R}(\omega_N) \hat{\rho}_{N-1}(\omega_1, \dots, \omega_{N-1}) \\ &= \mathcal{R}(\omega_N) \dots \mathcal{R}(\omega_1) \hat{\rho}_0,\end{aligned}\quad (13)$$

where the variables  $K$  and  $p$  have been omitted for clarity reasons. Since the  $\omega_j$ 's are statistically independent of each other and are identically distributed, one obtains:

$$\hat{\rho}_N = \bar{\mathcal{R}}^N \hat{\rho}_0 \quad (14)$$

where  $\bar{\mathcal{R}}$  is the statistical average of the evolution operator  $\mathcal{R}$  over the random angle  $\omega = \theta$  or  $\xi$  (the other angle being fixed to its Hadamard value):

$$\begin{aligned}\bar{\mathcal{R}}(K, p, \sigma) &= \int_{I_\sigma} \mathcal{R}(\omega, K, p) p_\sigma(\omega) d\omega \\ \bar{\mathcal{R}}(K, p, \sigma) &= \int_{I_\sigma} \mathcal{R}(\omega, K, p) \frac{1}{\sigma} d\omega.\end{aligned}\quad (15)$$

The average evolution  $\bar{\mathcal{R}}$  is thus a function of  $(K, p)$  and of the noise parameter  $\sigma$  and can be computed analytically from (11). It determines the evolution of the average density operator completely and, therefore, the average transport. Since everything that follows pertains only to the average transport, we simplify the notation by dropping the bar on the letter  $\rho$  and the density operator of the averaged transport will now be designated simply by  $\rho$ .

A direct computation from (11) leads to the following exact expressions for the components of  $\bar{\mathcal{R}}$  in the basis  $\{v_1, v_2, v_3, v_4\}$  for case 1 (random electric field) and case 2 (random gravitational field):

$$\bar{\mathcal{R}}^e(K, p, \sigma) = \frac{1}{2} \begin{bmatrix} e^{2iK} & \text{sinc}(\sigma/2) e^{-2ip} & \text{sinc}(\sigma/2) e^{+2ip} & e^{-2iK} \\ \text{sinc}(\sigma/2) e^{2iK} & -\text{sinc}(\sigma) e^{-2ip} & e^{+2ip} & -\text{sinc}(\sigma/2) e^{-2iK} \\ \text{sinc}(\sigma/2) e^{2iK} & e^{-2ip} & -\text{sinc}(\sigma) e^{+2ip} & -\text{sinc}(\sigma/2) e^{-2iK} \\ e^{2iK} & -\text{sinc}(\sigma/2) e^{-2ip} & -\text{sinc}(\sigma/2) e^{+2ip} & e^{-2iK} \end{bmatrix}, \quad (16)$$

and

$$\bar{\mathcal{R}}^g(K, p, \sigma) = \frac{1}{2} \begin{bmatrix} e^{2iK} & \text{sinc}(\sigma) e^{-2ip} & \text{sinc}(\sigma) e^{+2ip} & e^{-2iK} \\ \text{sinc}(\sigma) e^{2iK} & -e^{-2ip} & e^{+2ip} & -\text{sinc}(\sigma) e^{-2iK} \\ \text{sinc}(\sigma) e^{2iK} & e^{-2ip} & -e^{+2ip} & -\text{sinc}(\sigma) e^{-2iK} \\ e^{2iK} & -\text{sinc}(\sigma) e^{-2ip} & -\text{sinc}(\sigma) e^{+2ip} & e^{-2iK} \end{bmatrix}, \quad (17)$$

It proves convenient for all subsequent computations to change basis in  $\rho$  space and introduce the new vectors  $u_1 = v_1 + v_4$ ,  $u_2 = v_1 - v_4$ ,  $u_3 = v_2 + v_3$ ,  $u_4 = v_2 - v_3$ . In this new basis, the components of  $\bar{\mathcal{R}}^e(K, p, \sigma)$  and  $\bar{\mathcal{R}}^g(K, p, \sigma)$  read:

$$\bar{\mathcal{R}}^e(K, p, \sigma) = \begin{bmatrix} \cos(2K) & i \sin(2K) & 0 & 0 \\ 0 & 0 & \text{sinc}(\sigma/2) \cos(2p) & -i \text{sinc}(\sigma/2) \sin(2p) \\ i \text{sinc}(\sigma/2) \sin(2K) & \text{sinc}(\sigma/2) \cos(2K) & \frac{(1-\text{sinc}(\sigma))}{2} \cos(2p) & i \frac{(\text{sinc}(\sigma)-1)}{2} \sin(2p) \\ 0 & 0 & i \frac{(1+\text{sinc}(\sigma))}{2} \sin(2p) & -\frac{(\text{sinc}(\sigma)+1)}{2} \cos(2p) \end{bmatrix}, \quad (18)$$

$$\bar{\mathcal{R}}^g(K, p, \sigma) = \begin{bmatrix} \cos(2K) & i \sin(2K) & 0 & 0 \\ 0 & 0 & \text{sinc}(\sigma) \cos(2p) & -i \text{sinc}(\sigma) \sin(2p) \\ i \text{sinc}(\sigma) \sin(2K) & \text{sinc}(\sigma) \cos(2K) & 0 & 0 \\ 0 & 0 & i \sin(2p) & -\cos(2p) \end{bmatrix}, \quad (19)$$

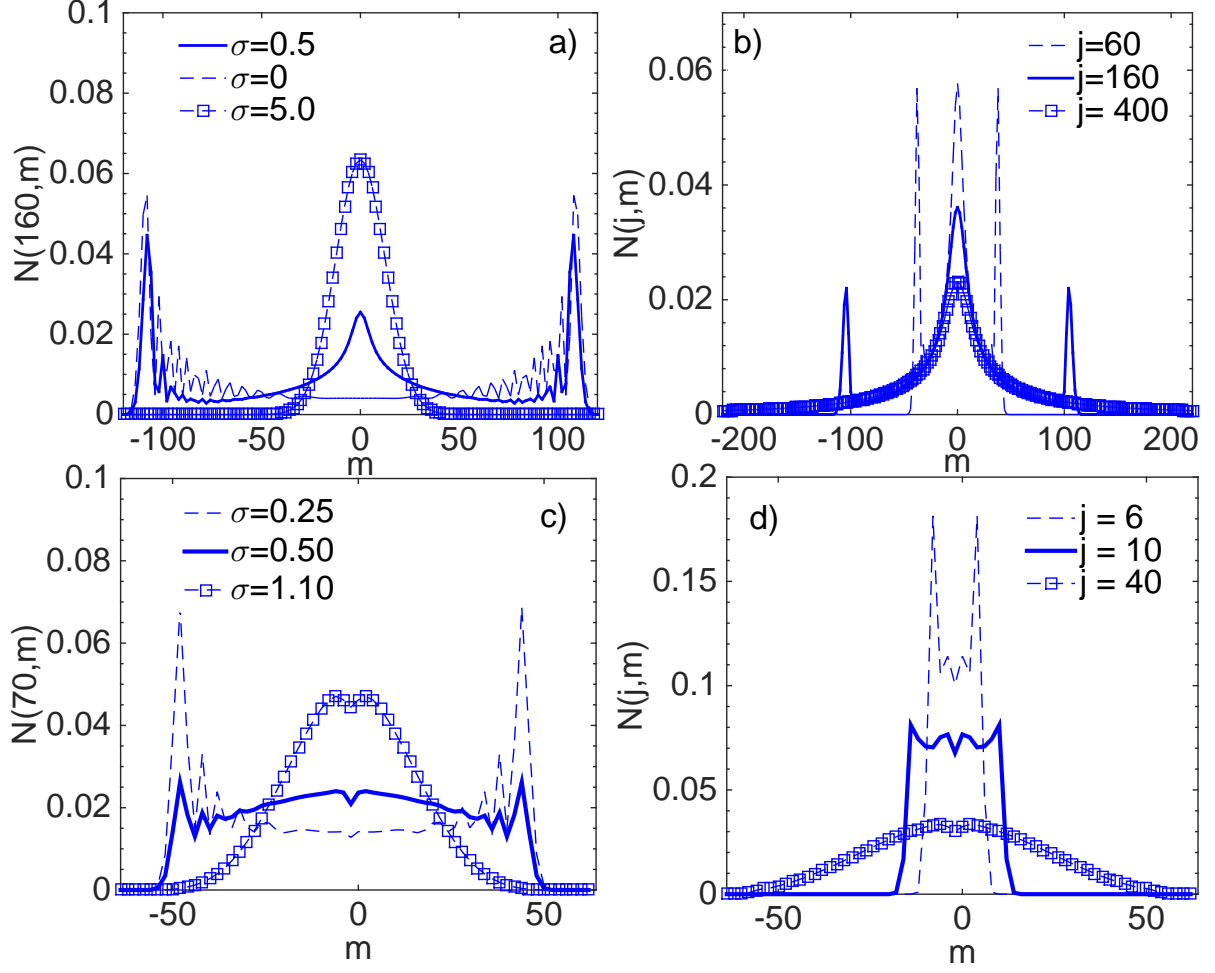


FIG. 1: (Color online) (left) Probability profile of the average transport in a random  $\xi$ -field (a) and  $\theta$ -field (c) vs grid point  $m$  at  $j = 160$  (a) and  $j = 70$  (c) for different values of the noise parameter  $\sigma$ . (right) Probability profile of the average transport in a random  $\xi$ -field (b) and  $\theta$ -field (d) vs grid point  $m$  for  $\sigma = 0.5$  (b) and  $\sigma = 0.8$  (d) at different time steps. Square marker represents fully decoherent regime, solid line the intermediate regime and dashed line indicates a fully coherent state.

We choose as initial condition the pure state defined by  $\Psi_{j=0,m=0} = (b_L + ib_R)/\sqrt{2}$  and  $\Psi_{j=0,m} = 0$  if  $m \neq 0$ . This state corresponds to the density operator  $\rho_{j=0,m=0,m'=0} = (b_L \otimes b_L + b_R \otimes b_R + i(b_R \otimes b_L - b_L \otimes b_R))/2 = (u_1 - iu_4)/2$  and  $\rho_{j=0,m,m'} = 0$  if  $m \neq 0$  or  $m' \neq 0$ . In Fourier space,  $\hat{\rho}_{j=0}(K, p) = (u_1 - iu_4)/2$  for all  $K$  and  $p$ .

For any realization of the noise *i.e.* for any given value of  $\omega$ , the initial pure state evolves by the DTQW into a pure state. But the average evolutions described by  $\bar{\mathcal{R}}^e$  and  $\bar{\mathcal{R}}^g$  both transform the initial pure state into a superposition. However, the average transport is symmetrical around the origin, as is the classical Hadamard walk generated from the same initial condition.

Let us finally stress that, contrary to the operator  $\mathcal{R}$  governing the unaveraged transport, the operators  $\mathcal{R}^{e/g}$  governing the averaged transport are *not* unitary. This loss of unitarity generates qualitative differences between the unaveraged and the averaged transport. In particular, the averaged transport loses quantum coherence and is asymptotically diffusive. These two important consequences of the averaging process are analyzed in the remaining sections of this article.

#### IV. QUALITATIVE DESCRIPTION OF AVERAGE TRANSPORT

Typical density profiles of the average transport are shown in Figure 1 for random gravitational and random electric fields. For small enough values of the noise parameter  $\sigma$ , the average transport behaves at short times like the Hadamard walk and is ballistic. Ballistic behavior then gradually disappears and is replaced by diffusive behavior. For

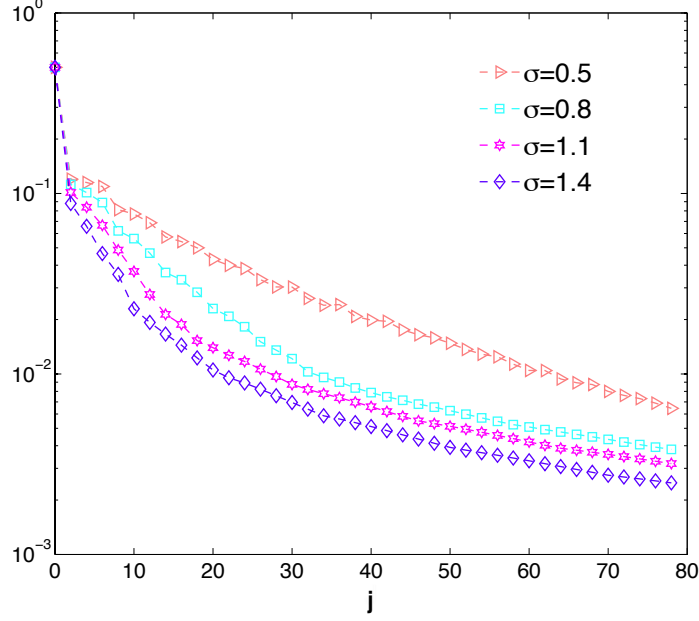


FIG. 2: Log-lin plot of time evolution of the spin coherence  $C_j$  in a random  $\theta$ -field for various values of the noise parameter  $\sigma$ .

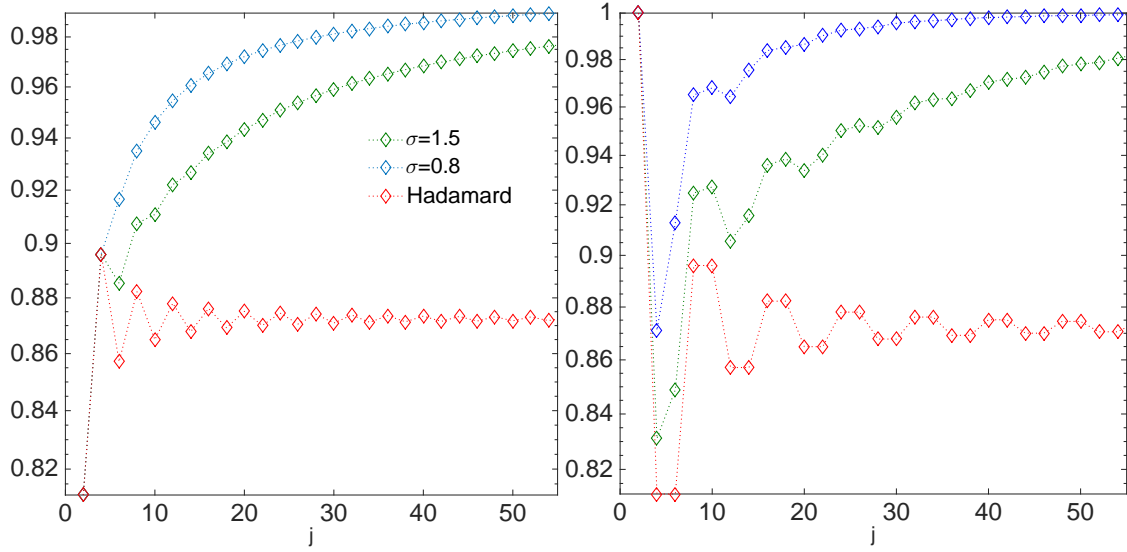


FIG. 3: Log-lin plot of time evolution of Shannon entanglement entropy  $S_r$  compared to the Shannon entanglement entropy of the average transport in a random  $\theta$ -field (left) and  $\xi$ -field (right)

larger values of  $\sigma$ , ballistic behavior is replaced, even at short times, by diffusive behavior. Note that the Gaussian-like form of the asymptotic density profiles presents a central dip when the DTQWs interact with random gravitational fields, but presents a central cusp when the DTQWs interact with random electric fields.

Asymptotically, DTQWs in electric and gravitational fields which are random in time thus behave like classical random walks. This means that the randomness in the fields prompts the DTQWs to loose coherence. This can be confirmed by considering the spin coherence defined by

$$C_j = \max_{m,m'} |\rho_{j,m,m'}^{LR}|. \quad (20)$$

Figure 2 displays the typical time-evolution of the spin coherence for various values of the noise parameter  $\sigma$ . These results confirm that the average transport loses spin coherence and that a higher value of  $\sigma$  leads to a quicker loss of spin coherence.

A brief comment on spatial coherence is in order. The retained initial condition vanishes everywhere except at  $m = m' = 0$ . If one prefers, the Fourier transform of the initial density operator is flat in both  $K$  and  $p$  space. There is thus initially no spatial coherence. As time increases, the Fourier transform  $\hat{\rho}(K, p)$  of the density operator  $\rho$  becomes non flat in both  $K$  and  $p$  (see for example the asymptotic form (21) of  $\hat{\rho}$ ). In other words, each  $K$ -mode acquires spatial coherence. But  $\tilde{\rho}(p) = \sum_K \hat{\rho}(K, p)$  remains flat in  $p$  (data not shown) i.e. there is no *total* gain of spatial coherence.

The entanglement of the averaged dynamics can also be quantified by the Shannon entropy  $S_r$  of the reduced density operator  $\rho_r$  in spin space. To be precise [29–31],  $\rho_r = \sum_m \rho_{m, m'=m}$  and the Shannon entropy  $S_r = -\text{tr}(\rho_r \log(\rho_r))$ . The time-evolution of  $S_r$  is presented in Figure 3, together with the entanglement entropy of the pure Hadamard walk with the same initial condition, which admits 0.872 as asymptotic value [32]. The increase in  $S_r$  signals the loss of coherence and the figure confirms that this decoherence by noise gets more effective as  $\sigma$  increases.

The scaling of the decoherence time for small values of  $\sigma$  can be evaluated by the following reasoning. As previously explained, the operator  $\bar{\mathcal{R}}^{e/g}$  completely controls the average dynamics. For  $\sigma = 0$ , there is no noise and the DTQW never decoheres i.e. the decoherence time is infinite. The first non-vanishing terms in the expansion of  $\bar{\mathcal{R}}^{e/g}$  around  $\sigma = 0$  are of second order in  $\sigma$ . Thus, per time step, the effect of the noise on the DTQW is of order  $\sigma^2$  for small enough values of  $\sigma$ . The typical decoherence time therefore scales as  $\sigma^{-2}$  for small values of  $\sigma$ .

The next section, together with the appendices, provides an analytical investigation of how coherence is lost. In particular, the asymptotic form of the density operator is computed exactly. The corresponding density is Gaussian, which confirms that the DTQW behaves asymptotically like a classical random walk. Also, the asymptotic density operator is proportionnal to  $u_1 = v_1 + v_4 = b_L \otimes b_L + b_R \otimes b_R$ . This proves that the spin coherence, which measures the amplitude of the  $b_L \otimes b_R$  component, vanishes asymptotically, in accordance with Figure 2.

## V. QUANTITATIVE DESCRIPTION OF THE ASYMPTOTIC REGIME

### A. Central limit theorem

The average dynamics is entirely determined by the eigenvalues  $\lambda_r^{e/g}$  and corresponding eigenvectors  $w_r^{e/g}$ ,  $r = 1, 2, 3, 4$ , of the operators  $\bar{\mathcal{R}}^{e/g}$ . As evident from Figure 1, the density profiles of the average transport become larger and smoother with time. This suggests that the asymptotic dynamics can be understood by computing the eigenvalues and eigenvectors only for values of  $K$  much smaller than unity. The detailed analysis, though very instructive, is too involved to merit inclusion in the main body of this article and it is therefore presented in the Appendix. The main conclusion can be stated as follows.

**Theorem.** *Let  $K_j = K_*/\sqrt{j}$  where  $K_*$  is an arbitrary but  $j$ -independent wave number. The average density operator in Fourier space admits as the time  $j$  tends to infinity the following approximate asymptotic expression:*

$$\hat{\rho}_j^{e/g}(K_j, p) \sim \frac{1}{2} \exp\left(-\alpha^{e/g}(p, \sigma) j K_j^2\right) u_1 \quad (21)$$

where

$$\alpha^e(p, \sigma) = 2 \frac{3 + (\text{sinc}(\sigma))^2 + 2(\text{sinc}(\sigma/2))^2(1 + \text{sinc}(\sigma)) + 4\cos(2p)(\text{sinc}(\sigma) + (\text{sinc}(\sigma/2))^2)}{3 + (\text{sinc}(\sigma))^2 - 2(\text{sinc}(\sigma/2))^2(1 + \text{sinc}(\sigma)) + 4\cos(2p)(\text{sinc}(\sigma) - (\text{sinc}(\sigma/2))^2)} \quad (22)$$

and

$$\alpha^g(p, \sigma) = 2 \frac{1 + (\text{sinc}(\sigma))^2}{1 - (\text{sinc}(\sigma))^2}. \quad (23)$$

This result is a central limit theorem which proves that the asymptotic density operator is approximately Gaussian in  $K$ -space, with a typical width (in  $K$ -space) which decreases as  $j^{-1/2}$ , as in classical random walks and non quantum diffusions. Note that  $\alpha^g$  is actually independent of  $p$ .

One of the consequences of (21) is that the projection of  $\hat{\rho}_{j=J}(K_J, p)$  on the subspace spanned by  $(u_2, u_3, u_4)$  tends to zero. Remembering the expressions of the  $u_i$  in terms of  $b_L$  and  $b_R$ , this means that  $\hat{\rho}^{LR}$ ,  $\hat{\rho}^{LR}$  and  $\hat{\rho}^{LL} - \hat{\rho}^{RR}$  all tend to zero as  $J$  tends to infinity. The component along  $u_1$  coincides with  $\hat{\rho}^{LL} + \hat{\rho}^{RR}$  and determines the asymptotic density of the averaged walk after summation over  $p$  and Fourier transform over  $K$ .



## B. Asymptotic mean-square displacement

Let us now explicitly compute the asymptotic expression of the mean-square displacement  $\overline{m^2}^{e/g}$  in the special case of a random DTQW on the infinite line. Switching back the original spatial variables  $m$  and  $m'$  involves a double integration over  $K$  and  $p$ . The  $2D$  measure to be used in this integration is  $dkdk' = 2dKdp$ . The density  $N_{jm}^{e/g}$  at time  $j$  and point  $m$  is the trace over  $m' = m$  of the component of the density operator along the basis vector  $u_1$ . Expression (C10) for  $\hat{\rho}_j^{e/g}$  is only valid for  $K \ll 1$  (see the Appendix). But the functions  $\alpha^{e/g}(p, \sigma)$  are always non vanishing. The width  $\Delta K(j, p)$  of  $\hat{\rho}_j^{e/g}(K, p)$  in  $K$  thus scales as  $1/\sqrt{j}$  and tends to zero as  $j$  tends to infinity. Thus, for large enough  $j$ , the density and mean square displacement are given by:

$$N_{jm}^{e/g} = \frac{1}{4\pi^2} \int_{p=-\pi}^{\pi} dp \int_{K=-\pi}^{\pi} dK \exp\left(-\alpha^{e/g}(p, \sigma)jK^2\right) \exp(-iKm) \quad (24)$$

and

$$\overline{m^2}^{e/g}(j, \sigma) = \frac{1}{4\pi^2} \sum_{m \in \mathbb{Z}} m^2 \int_{p=-\pi}^{\pi} dp \int_{K=-\pi}^{\pi} dK \exp\left(-\alpha^{e/g}(p, \sigma)jK^2\right) \exp(-iKm). \quad (25)$$

Since the width  $\Delta K(j, p)$  of  $\hat{\rho}_{j,K,p}$  scales as  $1/\sqrt{j}$ , one can also replace all discrete summations over  $m$  by integrals over the real line, because  $\Delta K(j, p) \times \Delta x = 1/\sqrt{j} \times 1 \ll 1$  for large enough  $j$ . Indeed, a simple computation confirms that the integrated density  $\int_{\mathbb{R}} dm N_{jm}^{e/g}$  (with  $N_{jm}^{e/g}$  given by (24)) is equal to unity at all times  $j$ . Replacing in (25) the discrete summation over  $m$  by an integral delivers

$$\overline{m^2}^{e/g}(j, \sigma) = \frac{j}{\pi} \int_{-\pi}^{\pi} \alpha^{e/g}(p, \sigma) dp. \quad (26)$$

The computation of  $\overline{m^2}^g(j, \sigma)$  is trivial because  $\alpha^g(p, \sigma)$  does not depend on  $p$ . One finds

$$\overline{m^2}^g(j, \sigma) = 2D^g(\sigma)j \quad (27)$$

with

$$D^g(\sigma) = 2 \frac{1 + (\text{sinc}(\sigma))^2}{1 - (\text{sinc}(\sigma))^2}. \quad (28)$$

The exact expression for  $\overline{m^2}^e(j, \sigma)$  is more involved. A direct computation leads to:

$$\overline{m^2}^e(j, \sigma) = 2D^e(\sigma)j \quad (29)$$

with

$$D^e(\sigma) = \frac{2}{\text{sinc}(\sigma) - (\text{sinc}(\sigma/2))^2} \left( \left( \text{sinc}(\sigma) + (\text{sinc}(\sigma/2))^2 \right) + \frac{2(\text{sinc}(\sigma/2))^2 \left( (\text{sinc}(\sigma))^2 + 2\text{sinc}(\sigma) - 3 \right)}{s(\sigma)} \right) \quad (30)$$

with

$$s(\sigma) = \left[ \left( 3 + (\text{sinc}(\sigma))^2 - 2(\text{sinc}(\sigma/2))^2(1 + \text{sinc}(\sigma)) \right)^2 - 16 \left( \text{sinc}(\sigma) - (\text{sinc}(\sigma/2))^2 \right)^2 \right]^{1/2}. \quad (31)$$

In both electric and gravitational case, the asymptotic mean square displacement in physical space grows linearly in time, as for classical random walks and non quantum diffusions. The functions  $D^e$  and  $D^g$  are the asymptotic diffusion coefficients of the average transport. Both functions are strictly decreasing on  $(0, 2\pi)$ . Thus, decoherence occurs more rapidly as  $\sigma$  increases (see Section IV), but the asymptotic diffusion coefficients decrease with  $\sigma$ . We also note that  $D^g(\sigma) < D^e(\sigma)$  for all  $\sigma \in (0, 2\pi)$ .

Figure 4 shows the time-evolution of the relative difference between the diffusion coefficients computed from (28), (29) and the mean square displacement computed from numerical simulations for various values of  $\sigma$ . This figures clearly supports the analytical computation presented in this Section.



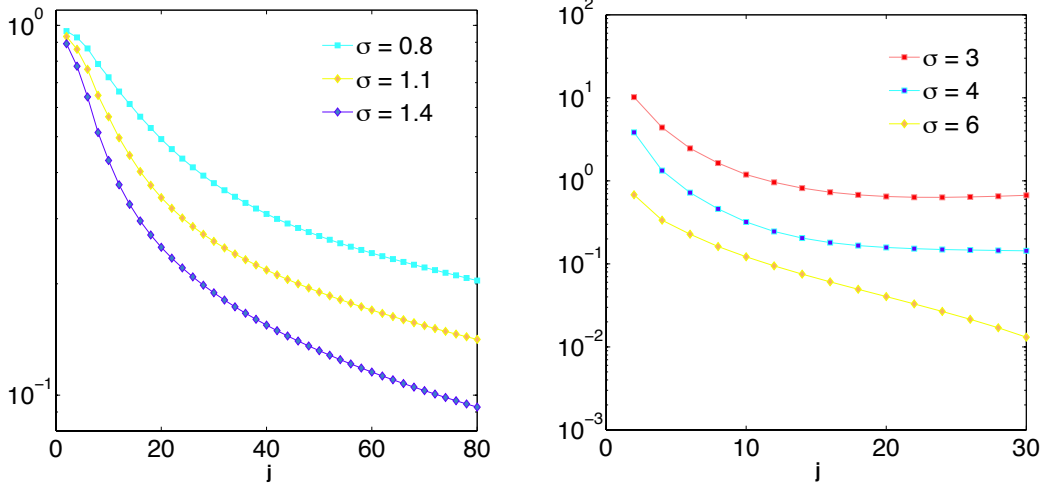


FIG. 4: Left (right) figure: time-evolution of the relative difference between the gravitational (electric) diffusion coefficients computed from numerical simulations and the exact analytical expressions.

## VI. CONTINUOUS LIMIT

The formal continuous limit of the original, unaveraged evolution equations (9) and (1) has already been considered in [21, 22] and coincides with the Dirac equation obeyed by a fermion minimally coupled to an electric field and/or a relativistic gravitational field. Let us now determine the formal continuous limit of the averaged evolution equations specified by the operators  $\bar{\mathcal{R}}^e$  and  $\bar{\mathcal{R}}^g$ .

As shown and discussed in [21, 22] for the unaveraged evolution equations, the object which admits a continuous limit for  $\theta = \theta_H$  or  $\xi = \xi_H$  is not the original walk, but the walk derived from it by keeping only one time step out of two [49]. We thus search for the continuous limit of the following discrete equations:

$$\hat{\rho}_{j+2}(K, p) = \left( \bar{\mathcal{R}}^{e/g}(K, p, \sigma) \right)^2 \hat{\rho}_j(K, p). \quad (32)$$

To be specific, we restrain  $j$  to uneven positive integer values and decide to work on the infinite line, so that  $K$  and  $p$  take all values in  $(-\pi, +\pi)$ .

We now suppose that, for all uneven  $j = 2r + 1$ ,  $\rho_{j=2r+1, m, m'}$  (resp.  $\hat{\rho}_{j=2r+1}(K, p)$ ) is the value taken by a certain function  $\rho$  (resp.  $\hat{\rho}$ ) at ‘time’  $t_r = r$  and positions  $x_m = m$  and  $x_{m'} = m'$  (resp. momenta  $K$  and  $p$ ). Roughly speaking, the continuous limit refers to situations where the function  $\rho$  (resp.  $\hat{\rho}$ ) varies only little during one time step  $t_{r+1} - t_r = 1$ . A necessary and sufficient condition for this to be realized is that  $(\bar{\mathcal{R}}^{e/g}(K, p, \sigma))^2$  be close to unity. Direct inspection reveals that this transcribes into  $\sigma \ll 1$ ,  $K \ll 1$  and  $p \ll 1$ . The last two conditions mean that  $\rho$  has characteristic spatial variation scales much larger than the distance  $m + 1 - m = 1$  between adjacent grid points and the first condition states that the noise amplitude is small. Note that  $K$ ,  $p$  and  $\sigma$  are *a priori* independent infinitesimal quantities. In particular, there is no reason why  $K$  and  $p$  should be of the same order of magnitude.

The formal continuous limit is then obtained by expanding  $(\bar{\mathcal{R}}^{e/g}(K, p, \sigma))^2$  around  $K = 0$ ,  $p = 0$ ,  $\sigma = 0$  and by replacing  $\hat{\rho}_{j+2} - \hat{\rho}_j$  by  $\partial_t \hat{\rho}$ . One thus gets equations of the form :

$$\partial_t \hat{\rho}(t, K, p) = \left( \mathcal{S}^{e/g}(K, p, \sigma) - 1 \right) \hat{\rho}(t, K, p) \quad (33)$$

where, for example,

$$\mathcal{S}^g(K, p, \sigma) = \begin{bmatrix} 1 - 4K^2 & 2iK & 2iK(1 - \frac{p^2}{2})(1 - \frac{\sigma^2}{6}) & 2Kp(1 - \frac{\sigma^2}{6}) \\ 2iK(1 - \frac{p^2}{2})(1 - \frac{\sigma^2}{3}) & (1 - 4K^2)(1 - \frac{p^2}{2})(1 - \frac{\sigma^2}{3}) & p^2(1 - \frac{\sigma^2}{6}) & 2ip(1 - \frac{\sigma^2}{6}) \\ 2iK(1 - \frac{\sigma^2}{6}) & -4K^2(1 - \frac{\sigma^2}{6}) & (1 - 4K^2)(1 - \frac{p^2}{2})(1 - \frac{\sigma^2}{3}) & -ip(1 - 4K^2)(1 - \frac{\sigma^2}{3}) \\ -2Kp(1 - \frac{\sigma^2}{6}) & ip(1 - 4K^2)(1 - \frac{\sigma^2}{6}) & -ip & 1 - 4p^2 \end{bmatrix} \quad (34)$$

at second order in all three independent infinitesimals  $K$ ,  $p$  and  $\sigma$ . These equations can be translated into physical space by remembering that  $-iK$  and  $-ip$  are the Fourier representations of  $\partial_X$  and  $\partial_y$  where  $X = (x + x')/2$  and  $y = x' - x$ .

The analysis presented in Sections III and IV above has been carried out with an initial condition which spreads over the whole  $K$ - and  $p$ -ranges. The resulting density operator does localize in time around  $K = 0$ , but it never localizes around  $p = 0$  and remains spread in  $p$ -space. The continuous limit thus cannot be used to recover the results of Section IV. As can be checked directly from (34), the continuous limit equations nevertheless predict diffusive behavior if  $K$  is much lower than both  $p$  and  $\sigma$ . A systematic study of the continuous limit dynamics for various scaling laws obeyed by  $K$ ,  $p$  and  $\sigma$  falls outside the scope of this article and will be presented elsewhere.

## VII. CONCLUSION

We have studied two families of DTQWs which can be considered as simple models of quantum transport of a Dirac fermion in random electric or gravitational fields. We have proven analytically and confirmed numerically that randomness of the fields in time leads on average to decoherence of the walks. The asymptotic average transport is thus diffusive and we have computed exactly the diffusion coefficients. We have also obtained and discussed the continuous limit of the model.

A few words about the loss of coherence in DTQWs may prove useful at this point. Pure, deterministic DTQWs are standard quantum systems in the sense that their time-evolution is unitary. They thus never loose coherence nor do they exhibit diffusive behavior. As with any quantum system, the loss of coherence in DTQWs is induced by the so-called interaction with an environment. There are essentially two ways to model this interaction. The first one is to start from the unitary evolution of the density operator and to modify this unitary evolution into a non-unitary one by introducing so-called projector or measurement operators [23–25, 33]. The second way of introducing decoherence is the one followed in this article. It consists in introducing some randomness in the parameters of the DTQW and in averaging over this randomness [26–28, 34, 35]. Contrary to the unaveraged density operator, the averaged density operator then follows a non-unitary evolution and this breakdown of unitarity induces the loss of coherence and the asymptotic diffusive behavior displayed by the averaged transport.

The results of this article constitute/are an addition to the already extensive literature dealing with the asymptotic behavior of DTQWs and CTQWs. Standard deterministic QWs are famous for typically exhibiting asymptotic ballistic behavior. But diffusive and anomalous diffusive asymptotic behavior have also been observed [36, 37], as well as localization [35, 38, 39] and soliton-like structures [34].

Let us conclude by listing a few natural extensions of this work. The random artificial gauge fields considered in this article have two main characteristics: they depend only on time and the associated mean fields vanish [40]. One should therefore extend the analysis presented above to situations where the mean fields do not vanish and where the artificial gauge fields depend not only on time, but also on position. In particular, the continuous limit equation derived in Section VI is markedly different from both the Caldeira-Leggett [41, 42] and the relativistic Kolmogorov equation describing relativistic stochastic processes [43–45]. Indeed, because the random fields depend only on time, the dynamics considered in this article does not couple different  $(K, p)$ -modes, but these are coupled in both the Caldeira-Leggett and the relativistic Kolmogorov equation. Considering DTQWs coupled to artificial gauge fields which also depend randomly on position should therefore lead to master equations closer to the the Caldeira-Leggett and the Kolmogorov models. Moreover, cases where both electric and gravitational fields vary randomly are certainly worth investigating.

Finally, at least some DTQWs in two spatial dimensions can be considered as models of quantum transport in electromagnetic fields [46]. The analysis presented in this article should therefore be repeated in higher dimensions to include random magnetic fields [47] and evaluate their effects on spintronics.

## Appendix A: Interpretation in terms of artificial gauge fields

It has been proven in [20–22] that quantum walks in  $(1 + 1)$  dimensional space-times can be viewed as modeling the transport of a Dirac fermion in artificial electric and gravitational fields generated by the time-dependance of the angles  $\theta$  and  $\xi$ . We recall here some basic conclusions obtained in [20–22] and also offer new developments useful in interpreting the results of the present article.

The DTQWs defined by (1) are part of a larger family whose dynamics reads:

$$\begin{bmatrix} \psi_{j+1,m}^L \\ \psi_{j+1,m}^R \end{bmatrix} = \tilde{B}(\theta_{j,m}, \xi_{j,m}, \zeta_{j,m}, \alpha_{j,m}) \begin{bmatrix} \psi_{j,m+1}^L \\ \psi_{j,m-1}^R \end{bmatrix}, \quad (\text{A1})$$

where

$$\tilde{B}(\theta, \xi, \zeta, \alpha) = e^{i\alpha} \begin{bmatrix} e^{i\xi} \cos \theta & e^{i\zeta} \sin \theta \\ -e^{i\zeta} \sin \theta & e^{-i\xi} \cos \theta \end{bmatrix}. \quad (\text{A2})$$

The walks in this larger family are characterized by three time- and space-dependent Euler angles  $(\theta, \xi, \zeta)$  and by a global, also time- and space-dependent phase  $\alpha$ . They have been shown to model the transport of Dirac fermions in artificial electric and relativistic gravitational fields generated by the time-dependence of the three Euler angles and of the global phase. In a  $(1+1)$  dimensional space-time, an electric field derives from a 2-potential  $A_{j,m} = (V_{j,m}, \mathcal{A}_{j,m})$  and a relativistic gravitational field is represented by 2D metrics  $G_{j,m}$ . The walks considered in this article correspond to

$$\begin{aligned} \xi &= \frac{\pi}{2} + \bar{\xi}_j \\ \theta &= \frac{\pi}{4} + \bar{\theta}_j \\ \alpha &= \frac{\pi}{2} + \bar{\alpha} \\ \zeta &= 0 \end{aligned} \quad (\text{A3})$$

where  $\bar{\xi}_j$  and  $\bar{\theta}_j$  are random variables which depend on the time  $j$  and  $\bar{\alpha} = 3\pi/2$ . According to [22], these walks model the transport of a Dirac fermion in an electric field generated by the 2-potential

$$A_j = (V_j, \mathcal{A}_j) = (\bar{\alpha}_j, -\bar{\xi}_j) = (\pi/2, -\bar{\xi}_j) \quad (\text{A4})$$

and in a gravitational field characterized by the metric

$$G_j = \text{diag}(1, -\cos^{-2}(\theta_j)). \quad (\text{A5})$$

Since relativistic gravitational fields are represented by space-time metrics [48], making the angle  $\theta$  a time-dependent random variable is equivalent to imposing a time-dependent random gravitational field. To better understand the electric aspects of the problem, let us recall that the DTQWs defined by (A1) exhibit the following exact discrete gauge invariance [22]:

$$\begin{aligned} \Psi'_{j,m} &= \Psi_{j,m} e^{-i\phi_{j,m}} \\ \xi'_{j,m} &= \xi_{j,m} + \delta_{j,m} \\ \theta'_{j,m} &= \theta_{j,m} \\ \alpha'_{j,m} &= \alpha_{j,m} + \frac{\sigma_{j,m}}{2} \\ \zeta'_{j,m} &= \zeta_{j,m} - \delta_{j,m} \end{aligned} \quad (\text{A6})$$

where

$$\begin{aligned} \sigma_{j,m} &= \phi_{j,m+1} + \phi_{j,m-1} - 2\phi_{j,m} \\ \delta_{j,m} &= \frac{\phi_{j,m+1} - \phi_{j,m-1}}{2} \end{aligned} \quad (\text{A7})$$

and  $\phi$  is an arbitrary time- and space-dependent phase shift. Let us now define a new quantity  $E_{j,m}$  by

$$E_{j,m} = -(\mathcal{D}_s V)_{j,m} + (\mathcal{D}_t \mathcal{A})_{j,m} \quad (\text{A8})$$

where the actions of the operators  $\mathcal{D}_s$  and  $\mathcal{D}_t$  on an arbitrary time- and space-dependent quantity  $u_{j,m}$  are

$$(\mathcal{D}_s u)_{j,m} = \frac{u_{j,m+1} - u_{j,m-1}}{2} \quad (\text{A9})$$

and

$$(\mathcal{D}_t u)_{j,m} = \frac{2u_{j+1,m} - u_{j,m+1} - u_{j,m-1}}{2}. \quad (\text{A10})$$

The operators  $\mathcal{D}_s$  and  $\mathcal{D}_t$  are discrete counterparts of space- and time-derivatives. It is straightforward to check that the quantity  $E_{j,m}$  is gauge invariant and coincides, in the continuous limit, with the standard electric field  $E(t, x)$ , defined by  $E(t, x) = -\partial_x V + \partial_t \mathcal{A}$ . The quantity  $E_{j,m}$  is thus a *bona fide* electric field in discrete space-time. For the DTQWs considered in this article, this electric field depends only on the time  $j$  and is related to the angle  $\bar{\xi}$  by  $E_j = -(\bar{\xi}_{j+1} - \bar{\xi}_j)$ . Making this angle a time-dependent random variable is thus equivalent to imposing a random electric field.

## Appendix B: Asymptotic computation of the eigenvalues and eigenvectors of the averaged transport operators

Let us here compute the eigenvalues  $\lambda_r^{e/g}$  and eigenvectors  $w_r^{e/g}$ ,  $r = 1, 2, 3, 4$  only for values of  $K$  much smaller than unity. We do not perform an expansion in  $p$  because the initial condition is uniform in  $p$  and the average evolution does not localize the density operator around  $p = 0$ . Indeed, the initial condition is localized at  $x' = x$  i.e. does not exhibit any spatial correlation and the dynamics does not create spatial correlations.

The second order expansions of the operators  $\bar{\mathcal{R}}^e$  and  $\bar{\mathcal{R}}^g$  in  $K$  read:

$$\bar{\mathcal{R}}_2^e(K, p, \sigma) = \begin{bmatrix} 1 - 2K^2 & 2iK & 0 & 0 \\ 0 & 0 & \frac{\text{sinc}(\sigma/2) \cos(2p)}{2} & -i \frac{\text{sinc}(\sigma/2) \sin(2p)}{2} \\ 2i \text{sinc}(\sigma/2) K & \text{sinc}(\sigma/2) (1 - 2K^2) & \frac{(1 - \text{sinc}(\sigma)) \cos(2p)}{2} & i \frac{(\text{sinc}(\sigma) - 1) \sin(2p)}{2} \\ 0 & 0 & i \frac{(1 + \text{sinc}(\sigma)) \sin(2p)}{2} & -\frac{(\text{sinc}(\sigma) + 1) \cos(2p)}{2} \end{bmatrix}, \quad (\text{B1})$$

and

$$\bar{\mathcal{R}}_2^g(K, p, \sigma) = \begin{bmatrix} 1 - 2K^2 & 2iK & 0 & 0 \\ 0 & 0 & \text{sinc}(\sigma) \cos(2p) & -i \text{sinc}(\sigma) \sin(2p) \\ 2i \text{sinc}(\sigma) K & \text{sinc}(\sigma) & 0 & 0 \\ 0 & 0 & i \sin(2p) & -\cos(2p) \end{bmatrix}. \quad (\text{B2})$$

For  $K = 0$ , these two matrices are both block diagonal and we write  $\bar{\mathcal{R}}_2^{e/g}(K = 0, p, \sigma) = \text{diag}(1, M^{e/g}(p, \sigma))$ , where  $M^{e/g}(p, \sigma)$  are  $3 \times 3$  matrices acting in the space spanned by  $(u_2, u_3, u_4)$ . The matrices  $\bar{\mathcal{R}}_2^{e/g}(K = 0, p, \sigma)$  share  $u_1$  as common eigenvector, which we identify as  $w_1^{e/g}(K = 0, p, \sigma)$ ; the associated eigenvalue is  $\lambda_1^{e/g}(K = 0, p, \sigma) = 1$ . The other eigenvectors and eigenvalues, at zeroth order in  $K$ , are those of  $M^{e/g}(p, \sigma)$ . These eigenvalues can be computed analytically by solving the third-order characteristic polynomials associated to these matrices. The explicit expressions of these eigenvalues are quite involved and need not be replicated here. What is important is how the moduli of these eigenvalues compares to unity. Direct inspection reveals that the moduli of all three  $\lambda_r^e(0, p, \sigma)$ ,  $r = 2, 3, 4$  are strictly inferior to unity if  $\sigma$  is not vanishing. The same goes for all three eigenvalues in the gravitational case, except for one of them which reaches  $\pm 1$  independantly of  $\sigma$  for  $p = \pm\pi$  and is also equal to  $+1$  for  $p = 0$ ; the eigenspaces corresponding to  $\lambda_4^g(\pm\pi, \sigma)$  and  $\lambda_4^g(0, \sigma)$  are identical and generated by  $u_4$ , which we choose as  $w_4^g(p = \pm\pi, \sigma) = w_4^g(0, \sigma)$ . For other values of  $p$ , the eigenvalue  $\lambda_4^g(p, \sigma)$  and the eigenvector  $w_4^g(p, \sigma)$  are defined by continuity. All other eigenvectors need not be specified for what follows.

Let us now turn to non vanishing values of  $K$ . The characteristic polynomials of  $\bar{\mathcal{R}}_2^{e/g}(K, p, \sigma)$  contain terms of order 2 and 4 in  $K$ ; at lowest order in  $K$ , the corrections to the eigenvalues  $\lambda_j^{e/g}(K = 0, p, \sigma)$  thus scale generically as  $K^2$ . Let  $\lambda$  be the variable of the characteristic polynomials. At second order in  $K$ , the  $K$ -dependent correction to each of the zeroth order eigenvalues  $\lambda_r^{e/g}(K = 0, p, \sigma)$  can be found by expanding the characteristic polynomial of  $\bar{\mathcal{R}}_2^{e/g}(K, p, \sigma)$  at first order in  $(\lambda - \lambda_r^{e/g}(K = 0, p, \sigma))$  and by keeping only the terms scaling as  $K^2$ . This gives rational expressions for the corrections to the eigenvalues; these rational expressions can be further simplified by a final expansion around  $K = 0$  if  $p$  is treated as a finite, non infinitesimal quantity i.e.  $|K| \ll |p|$ . One then finds:

$$\lambda_1^{e/g}(K, p, \sigma) = 1 - \alpha^{e/g}(p, \sigma) K^2 + O(K^4) \quad (\text{B3})$$

with

$$\alpha^e(p, \sigma) = 2 \frac{3 + (\text{sinc}(\sigma))^2 + 2 (\text{sinc}(\sigma/2))^2 (1 + \text{sinc}(\sigma)) + 4 \cos(2p) (\text{sinc}(\sigma) + (\text{sinc}(\sigma/2))^2)}{3 + (\text{sinc}(\sigma))^2 - 2 (\text{sinc}(\sigma/2))^2 (1 + \text{sinc}(\sigma)) + 4 \cos(2p) (\text{sinc}(\sigma) - (\text{sinc}(\sigma/2))^2)} \quad (\text{B4})$$

and

$$\alpha^g(p, \sigma) = 2 \frac{1 + (\text{sinc}(\sigma))^2}{1 - (\text{sinc}(\sigma))^2}. \quad (\text{B5})$$

Note that  $\alpha^g$  is actually independent of  $p$ . Note also that the condition  $|K| \ll |p|$  does not hinder asymptotic computations, at least on the infinite line. Indeed, as time increases, the density operator becomes more and more localized around  $K = 0$ , but it does not localize in  $p$ -space [50]. If one works on the infinite line, both  $K$  and  $p$  are continuous variables and the localization of the density operator around  $K = 0$  implies that the size of the region in

$p$ -space where the condition  $|K| \ll |p|$  does not apply actually shrinks to zero with time. For dynamics taking place on a finite circle (finite value of  $M$ ), computations are a little more involved but can nevertheless be carried out. We feel a detailed analysis of the problem for finite values of  $M$  does not bring any valuable insight on interesting physics or mathematics, and we thus restrict the analytical discussion of the asymptotic dynamics to DTQWs on the infinite line, where expressions (B4) and (B5) suffice.

A direct computation shows that the corrections to the eigenvectors are first order in  $K$ . By convention, we fix to unity the value of the first component of  $w_1^{e/g}(K, p, \sigma)$  in the basis  $(u_1, u_2, u_3, u_4)$ . One thus gets for example

$$w_1^g(K, p, \sigma) = \{1, \frac{2iK \text{sinc}(\sigma)^2}{1 - \text{sinc}(\sigma)^2}, \frac{2iK \text{sinc}(\sigma)}{1 - \text{sinc}(\sigma)^2}, \frac{-2K \text{sinc}(\sigma) \tan(p)}{1 - \text{sinc}(\sigma)^2}\}. \quad (\text{B6})$$

The expression of  $w_1^e$  is substantially more complicated and need not be reproduced here.

### Appendix C: Asymptotic expression of the density operator in Fourier space

Let us now use the above results to determine the time evolution of the average density operator in both cases under consideration. The first step is to express the initial condition,  $\hat{\rho}_{j=0}(K, p) = (u_1 - u_4)/2$  for all  $(K, p)$ , as a linear combination of the eigenvectors  $w_r^{e/g}(K, p, \sigma)$ . We thus write, for  $a = 1, 2, 3, 4$

$$u_a = \sum_{r=1}^4 u_{ar}^{e/g}(K, p, \sigma) w_r^{e/g}(K, p, \sigma) \quad (\text{C1})$$

and, conversely,

$$w_r^{e/g}(K, p, \sigma) = \sum_{a=1}^4 w_{ra}^{e/g}(K, p, \sigma) u_a. \quad (\text{C2})$$

By the above discussion of the eigenvalues and eigenvectors of  $\bar{\mathcal{R}}_2^{e/g}$ , one has notably  $u_{11}^{e/g}(K, p, \sigma) = 1$ ,  $u_{1r}^{e/g}(K, p, \sigma) = O(K)$  for  $r = 2, 3, 4$ ,  $w_{11}^{e/g}(K, p, \sigma) = 1 + O(K)$ .

One then writes, for all  $K$  and  $p$ :

$$\hat{\rho}_{j=0}(K, p) = \frac{1}{2} \sum_{r=1}^4 \left( u_{1r}^{e/g}(K, p, \sigma) - u_{4r}^{e/g}(K, p, \sigma) \right) w_r^{e/g}(K, p, \sigma). \quad (\text{C3})$$

which leads to

$$\hat{\rho}_{j=J}(K, p) = \frac{1}{2} \sum_{r=1}^4 \left( \lambda_r^{e/g}(K, p, \sigma) \right)^J \left( u_{1r}^{e/g}(K, p, \sigma) - u_{4r}^{e/g}(K, p, \sigma) \right) w_r^{e/g}(K, p, \sigma) \quad (\text{C4})$$

or, expressing the eigenvectors  $w_r^{e/g}(K, p, \sigma)$  in terms of the original basis vectors  $(u_1, u_2, u_3, u_4)$ :

$$\hat{\rho}_{j=J}(K, p) = \frac{1}{2} \sum_{r=1}^4 \sum_{a=1}^4 \left( \lambda_r^{e/g}(K, p, \sigma) \right)^J \left( u_{1r}^{e/g}(K, p, \sigma) - u_{4r}^{e/g}(K, p, \sigma) \right) w_{ra}^{e/g}(K, p, \sigma) u_a \quad (\text{C5})$$

Now, for all  $r$ ,

$$\lambda_r^{e/g}(K, p, \sigma) / \lambda_1^{e/g}(K, p, \sigma) = \lambda_r^{e/g}(K = 0, p, \sigma) (1 + O(K^2)), \quad (\text{C6})$$

since  $\lambda_r^{e/g}(K = 0, p, \sigma) = 1$ . It follows that, for small enough  $K$ , the contributions to (C5) proportional to  $\left( \lambda_r^{e/g}(K, p, \sigma) \right)^J$  are much smaller than the contribution proportionnal to  $(\lambda_1^{e/g}(K, p, \sigma))^J$  for all values of  $p$  and  $\sigma$  such that  $|\lambda_r^{e/g}(K = 0, p, \sigma)| < 1$ . According to the above discussion, this is realized for all  $r \neq 1$  and for all values of  $p$  and  $\sigma$ , except in case 2 (random gravitational field) for  $r = 4$ ,  $p = \pm\pi$  or  $p = 0$  and all values of  $\sigma$ . What happens at  $p = \pm\pi$  has no incidence on the computation of the density operator in physical space. Indeed, for finite values of  $M$ , the maximum value  $p_{\max}$  of  $|p|$  is  $p_{\max} = (2M/(2M+1))\pi < \pi$ . Thus  $\pm\pi$  is only reached in the limiting case of infinite  $M$  i.e. for quantum walks in the infinite line. However,  $\pm p_{\max} = \pm\pi$  then only appear as upper and

lower bounds for integrals over  $p$ , and the values taken by  $\hat{\rho}(J, K, p)$  at points  $\pm\pi$  does not modify the values of the integrals. Moreover, all current computations are only valid for  $|p| \ll K$  and are thus a priori invalid for  $p = 0$ . What happens around  $p = 0$  has however no relevance to asymptotic computations on the infinite line because, as time increases, the density operator becomes more and more localized around  $K = 0$  (see discussion below (B5)).

For large enough  $J$  and small enough  $K$ , the double sum in (C5) thus simplifies into:

$$\hat{\rho}_{j=J}^{e/g}(K, p) = \frac{1}{2} \sum_{a=1}^4 \left( \lambda_1^{e/g}(K, p, \sigma) \right)^J \left( u_{11}^{e/g}(K, p, \sigma) - u_{41}^{g/e}(K, p, \sigma) \right) w_{1a}^{e/g}(K, p, \sigma) u_a \quad (C7)$$

Now,  $u_{11}^{e/g}(K, p, \sigma) = 1 + O(K)$ ,  $u_{41}^{g/e}(K, p, \sigma) = O(K)$ ,  $w_{11}^{e/g}(K, p, \sigma) = 1 + O(K)$  and  $w_{1b}^{e/g}(K, p, \sigma) = O(K)$  for  $b = 2, 3, 4$ . As far as orders of magnitude are concerned, equation (C7) gives:

$$\hat{\rho}_{j=J}^{e/g}(K, p) = \frac{1}{2} \left( 1 - \alpha^{e/g}(p, \sigma) K^2 \right)^J (1 + O(K)) u_1 + \sum_{b=2}^4 O(K) u_b. \quad (C8)$$

At lowest order in  $K$ ,  $(1 - \alpha^{e/g}(K, p, \sigma) K^2)^J = 1 - \alpha^{e/g}(K, p, \sigma) J K^2$ . We will now restrict the discussion to scales  $K$  and times  $J$  obeying  $JK^2 \gg K$  i.e.  $JK \gg 1$ . Note that the maximum spatial spread of  $\bar{\rho}$  at time  $J$  is  $L_{\max}(J) = 2J$ , so that the minimum value of  $K$  for which  $\hat{\rho}$  takes non negligible values at time  $J$  is of order  $K_{\min}(N) = 1/J$ . The condition  $JK \gg 1$  thus restricts the discussion to length scales much smaller than  $L_{\max}(N)$ . In particular, consider the time-dependent scale  $K_J = K_*/\sqrt{J}$ , where  $K_*$  is an arbitrary time-independent wave-vector. The wave-vector  $K_J$  obeys  $JK_J^2 = K_*^2 \gg K_J$  for sufficiently large  $J$ . Thus, the possible diffusive behavior of the averaged transport is encompassed by the present discussion.

With the above assumption, equation (C7) implies the following approximate but very simple expression for the long time (large  $J$ ) density operator in Fourier space:

$$\hat{\rho}_{j=J}^{e/g}(K, p) = \frac{1}{2} \left( 1 - \alpha^{e/g}(p, \sigma) K^2 \right)^J u_1. \quad (C9)$$

In particular, for  $K_J = K_*/\sqrt{J}$  (where  $K_*$  is an arbitrary but  $J$ -independent wave number) and large enough  $J$ ,

$$\hat{\rho}_{j=J}^{e/g}(K_J, p) = \frac{1}{2} \left( 1 - \alpha^{e/g}(p, \sigma) \frac{K_*^2}{J} \right)^J u_1 \sim \frac{1}{2} \exp \left( -\alpha^{e/g}(p, \sigma) K_*^2 \right) u_1 = \frac{1}{2} \exp \left( -\alpha^{e/g}(p, \sigma) J K_J^2 \right) u_1. \quad (C10)$$

This is the approximate expression for the asymptotic density operator presented in the main body of this article.

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as is the initial condition